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Theorem on the Maximum Temperature Gradient

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1. The intensification of the heat emission of a solid under invariant external conditions is usually attained by subdivision, by increasing the surface without increasing its weight, i.e., of the volume. Such an isochoric increase of the body surface causes a growth of its integral criterion of the form:

$$\text{Es} = \frac{l_S}{l_V} \quad ; \quad l_S = \begin{cases} P \\ \sqrt{S} \end{cases} \quad ; \quad l_V = \begin{cases} \sqrt{F} \\ \sqrt[3]{V} \end{cases} \tag{1}$$

It will be shown in the present paper that the growth of the thermal flow velocity:

$$Q' = Sq \quad {}^{\text{cal}}/\text{hr} \tag{2}$$

for an isochoric increase in the surface S , occurs simultaneously with a decrease in the density of the thermal flow speed:

$$q = -\lambda(\nabla\theta)_S = \alpha\theta_S(\tau) \tag{3}$$

on the body surface. Such a decrease in the density of the thermal flow speed (specific thermal flow) follows directly from the decrease in the mean value of the temperature gradient on the body surface $(\nabla\theta)_S$ as the Es kind of criterion increases or from the decrease in the mean temperature of the surface $\theta_S(\tau)$ together with the increase in the Es simplex. In other words:

$$\frac{\partial q}{\partial \text{ Es}} < 0 \tag{4}$$

2. It was indicated in [7], that the maximum temperature $\theta_m(\tau)$ of a body is higher, the higher the Es criterion of the body under the same remaining conditions.

This law is illustrated well in the graph of the work cited, which we borrowed from [3]. The recent experiments and computations of Veinik [1,2] confirmed this fact conclusively.

But if $\theta_m(\tau, Es)$ increases but $\theta_S(\tau, Es)$ decreases as Es increases, then it can be stated that the difference:

$$\Delta\theta = \theta_m(\tau, Es) - \theta_S(\tau, Es) \tag{5}$$

increases along with the Es criterion.

A warmest and a coldest point exist on the body surface at each moment τ. It is completely possible that such points with extremum temperatures are several. If two points of the surface - one with a maximum and the other with a minimum temperature - be joined by a line lying entirely on the surface, then, clearly, at least one point on this line will be at the temperature which is exactly equal to the average surface temperature $\theta_S(\tau)$ at a given time τ.

Let us join the warmest point of the body, which has the $\theta_m(\tau)$ temperature, to that surface point $P(\vec{r}_s)$ with a temperature equal to the average surface temperature $\theta_S(\tau)$, by a segment of length l. Understandably, a surface point close to $P(\vec{r}_s)$ can be taken where the temperature would be somewhat lower than $\theta_S(\tau)$.

Let us call the ratio:

$$\frac{\theta_m(\tau) - \theta(\vec{r}_s\tau)}{l} = |\nabla\theta|_{av} \tag{6}$$

the average value of the temperature gradient on this segment. According to the theorem of the mean, a point $P(\vec{r}_{av})$ must exist on this segment, where the absolute value of $|\nabla\theta(\vec{r}_{av}, \tau)|$ exactly equals $|\nabla\theta|_{av}$:

$$|\nabla\theta(\vec{r}_{av}, \tau)| = |\nabla\theta|_{av} \tag{7}$$

It was shown earlier [8] that, for a k-dimensional body:

(8) $$\frac{\partial\theta_V}{\partial Fo} = \zeta_V (\nabla\theta)_S Es_k^{k-1}$$

and, consequently, the average value of the temperature gradient on the body surface $(\nabla\theta)_S$ can be made less than any quantity assigned beforehand by a suitable choice of the Es criterion, i.e., by a suitable isochoric body deformation, for the same not-too-small Bi and Fo, and therefore, less than $|\nabla\theta|_{av}$, which increases along with Es.

The reservation relative to sufficiently large magnitudes of the Bi and Fo criteria is completely natural. If the intensity of the heat exchange is very small, which is characterized by small values of the Bi criterion, then the temperature field should be considered practically homogeneous and its gradients will generally be very small. If we consider the heat emission process at its very beginning, then the whole temperature field is slightly different from the original temperature distribution which is characterized by the absence of heat flow and, consequently, by the absence of temperature differences.

Therefore, it can be stated that, for not too small Bi and Fo values in bodies of complex enough shape, which is characterized by a high Es criterion, points exist within the body at which the absolute value of the temperature gradient is larger than the average value of the temperature gradient on the body surface:

(9) $$|\nabla\theta(\vec{r}_{av},\tau)| \quad (\nabla\theta)_S$$

This statement expresses the theorem on the maximum temperature gradient. Hence, there are points on bodies of complex configuration, where the specific thermal flow is larger than on the body surface, on the average.

But at the body temperature center, the gradient of a point where the temperature is a maximum must equal zero:

$$\nabla\theta(\vec{r}_m, \tau) = 0 \tag{10}$$

Because of the above-mentioned theorem, the value of the temperature gradient on certain lines issuing from the center is higher than on the surface, on the average. Consequently, the absolute value of the temperature gradient, on lines which join the temperature center of a body of complex shape to those points of the surface where the temperature equals the average surface temperature at a given time $\theta_S(\tau)$, increases from zero to a certain maximum as the center moves toward the surface and, furthermore, decreases to the $\nabla\theta(\vec{r}_S, \tau)$ value on the body surface. The curve, showing the variation of the temperature as a function of the distance from the body center, must have an inflection point on such a line.

3. The location of the point at which the absolute value of the temperature gradient is a maximum is determined from the system of equations:

$$\nabla\theta \frac{\partial \nabla\theta}{\partial u} = 0 \qquad (u = x, y, z) \tag{11}$$

for a three dimensional body and from the equations:

$$\begin{aligned} \frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2} + \frac{\partial\theta}{\partial y}\frac{\partial^2\theta}{\partial x \partial y} &= 0 \\ \frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x \partial y} + \frac{\partial\theta}{\partial y}\frac{\partial^2\theta}{\partial y^2} &= 0 \end{aligned} \tag{12}$$

for a two-dimensional.

The specific heat flow has its highest value at this point of the body. As the example cited below shows, this point moves, as the Bi criterion decreases, to the body boundary and will not, in general, be found within or on the body surface for small Bi values. In other words, the thermal flow $q = \lambda|\nabla t|$ increases monotonically, for small Bi, as the center of the body is left behind. The body projections are cooled more rapidly for large Bi than are the fundamental body mass, the temperature differences within the limits of these projections are insignificant and a certain part of the heat moves from the basic body mass to these projections because of the presence of the temperature differences between the basic body mass and the projection mass.

Let us clarify the location of the point through which the heat supply to the body projections occurs. These points determine, simultaneously, just what part of the body should be considered basic and what part as projection.

The motion of heat within a body is accomplished along thermal current lines which are orthogonal to the field isotherms. Since we consider a two-dimensional body as an example, then it is expedient for the later investigation, to carry out the plane problem. The differential equation of the thermal current line is:

$$\frac{dy}{dx}\frac{\partial\theta}{\partial x} - \frac{\partial\theta}{\partial y} = 0 \tag{13}$$

It is obtained from the orthogonality condition of the isotherm and the thermal current line. The length of a thermal current line element is:

$$ds = \sqrt{\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2}\ \frac{dx}{\frac{\partial\theta}{\partial x}} \tag{14}$$

Let us calculate the temperature drop along such a line. In connection with (14), we obtain after certain transformations:

$$\frac{\partial\theta}{\partial s} = \sqrt{\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2} \tag{15}$$

This result is natural because the highest temperature drop occurs along the current line and

$$\frac{\partial\theta}{\partial s} = |\nabla\theta| \tag{16}$$

Let us determine the derivative of the gradient along a length of the current line through the formula:

(17) $$\frac{\partial|\nabla\theta|}{\partial s} = \frac{\partial|\nabla\theta|}{\partial x}\frac{dx}{ds} + \frac{\partial|\nabla\theta|}{\partial y}\frac{dy}{ds}$$

Using (14), we will have:

(18) $$\frac{\partial^2\theta}{\partial x^2} = \frac{\frac{\partial^2\theta}{\partial x^2}\left(\frac{\partial\theta}{\partial x}\right)^2 + 2\frac{\partial^2\theta}{\partial x\partial y}\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial y} + \frac{\partial^2\theta}{\partial y^2}\left(\frac{\partial\theta}{\partial y}\right)^2}{\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2}$$

Therefore, at points where the thermal flow is a maximum, the condition holds:

(19) $$\frac{\partial^2\theta}{\partial x^2}\left(\frac{\partial\theta}{\partial x}\right)^2 + 2\frac{\partial^2\theta}{\partial x\partial y}\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial y} + \frac{\partial^2\theta}{\partial y^2}\left(\frac{\partial\theta}{\partial y}\right)^2 = 0$$

4. It was indicated above that for low values of the Fo criterion, which corresponds to the start of the process, i.e., to low values of the time, there are no large temperature gradients within the body and the thermal flow increases monotonically with withdrawal from the body center.

Consequently, the theorem on the maximum temperature gradient can appear to be incorrect in the irregular region stage and to be completely true in the regular region stage. The temperature of each body point varies exponentially in the regular region stage:

(20) $$\theta(\vec{r},\tau) = \theta(\vec{r},\tau_0)\exp[-m(\tau - \tau_0)]$$

where τ_0 is the moment the regular region starts, m is the cooling rate.

As is seen from (20), the isotherm shape in the regular region stage is invariant; the temperature of each isotherm decreases exponentially. The geometry of the temperature field in the regular region is independent of the time and only the geometric assumption, correct at the start of this stage $(\tau = \tau_0)$, is correct in the rest of the time $(\tau > \tau_0)$; consequently, we will consider the temperature field at the start of the regular region stage, when $\tau = \tau_0$.

As the object to which we apply our theory should be chosen a body from which a body with various shape criterion values can be obtained by a variation of the dimensions. It is most convenient to select a rectangular profile as such a body. Profiles with Es criterion values from 4.00 to ∞ can be obtained by varying the ratio of the sides of the rectangle.

The temperature of an infinitely long rectangular prism at time τ in the regular region stage is given by the dependence:

$$\theta(\vec{r},\tau) = \theta_{om} \cos \frac{\mu_1 x}{R_1} \cos \frac{\mu_2 y}{R_2} \exp\left[-m(\tau - \tau_o)\right] \tag{21}$$

where τ_o , as was indicated above, is the moment the regular region starts; R_1 and R_2 are half the sides of the rectangle, parallel to the Ox and Oy coordinate axes, $R_1 > R_2$; μ_1 and μ_2 are the least roots of the characteristic equations:

$$\mu \tan \mu = \frac{\alpha}{\lambda} R_n \qquad (n = 1,2) \tag{22}$$

θ_{om} is the temperature of the prism center at the start of the regular region

$$\theta(0,0,\tau_o) = \theta_{om} \tag{23}$$

The temperature of the medium is taken to be constant and equal to zero.

Let us clarify the location of the maximum temperature gradient depending on the $Bi_V = \frac{\alpha}{\lambda}\sqrt{R_1 R_2}$ criterion. In order to determine the point of the maximum gradient in correspondence with (12) and (21), the system of equations obtained is:

$$\begin{aligned} &\left(\frac{\mu_1}{R_1}\right)^3 \cos\frac{\mu_1 x}{R_1} \cos^2\frac{\mu_2 y}{R_2} \sin\frac{\mu_1 x}{R_1} - \left(\frac{\mu_2}{R_2}\right)^2 \frac{\mu_1}{R_1} \cos\frac{\mu_1 x}{R_1} \sin\frac{\mu_1 x}{R_1} \sin^2\frac{\mu_2 y}{R_2} = 0 \\ &- \left(\frac{\mu_1}{R_1}\right)^2 \frac{\mu_2}{R_2} \cos\frac{\mu_2 y}{R_2} \sin^2\frac{\mu_1 x}{R_1} \sin\frac{\mu_2 y}{R_2} + \left(\frac{\mu_2}{R_2}\right)^3 \cos^2\frac{\mu_1 x}{R_1} \cos\frac{\mu_2 y}{R_2} \sin\frac{\mu_2 y}{R_2} = 0 \end{aligned} \tag{24}$$

from which are found the coordinates of the point of the max. $|\nabla\theta|$:

$$(25)\qquad \begin{cases} \xi = \dfrac{x}{R_1} = \dfrac{1}{\mu_1} \arctan \dfrac{\mu_2 R_1}{R_2 \mu_1} \\ \eta = \dfrac{y}{R_2} = \dfrac{1}{\mu_2} \arctan \dfrac{\mu_1 R_2}{R_1 \mu_2} \end{cases}$$

or, taking (22) into account, these expressions can be transformed into:

$$(26)\qquad \begin{cases} \xi = \dfrac{1}{\mu_1} \arctan \dfrac{\tan \mu_1}{\tan \mu_2} \\ \eta = \dfrac{1}{\mu_2} \arctan \dfrac{\tan \mu_2}{\tan \mu_1} \end{cases}$$

For a square, for example, $R_1 = R_2 = R$, $\mu_1 = \mu_2 = \mu$ and

$$(27)\qquad \xi = \eta = \frac{\pi}{4\mu}$$

that is, the point on the square where the thermal flow is a maximum is certainly on its diagonal. The characteristic number μ for $Bi = \infty$ is $\frac{\pi}{2}$ and

$$(28)\qquad \xi = \eta = \frac{1}{2}$$

the max. $|\nabla\Theta|$ point is at the middle of the segment joining the center to a vertex of the square. For $Bi = \frac{\pi}{4}$

$$(29)\qquad \xi = \eta = 1$$

that is, the greatest heat flow is observed at the vertex of the square. Evidently, for $\frac{\alpha R}{\lambda} < \frac{\pi}{4}$, when $\mu < \frac{\pi}{4}$, the point with max. $|\nabla\Theta|$ will not, generally, exist in the square and the temperature gradient on each thermal current line will increase monotonically from zero at the center of the square to a corresponding magnitude on the boundary.

Below is given a table of values of the $\xi = \eta$ coordinates depending on the $Bi = \frac{\alpha R}{\lambda}$ quantity.

Table 1

Point of the max. $|\nabla\Theta|$ in the square

Bi	$\xi = \eta$	Bi	$\xi = \eta$
0.7854	1.000	10.0	0.551
0.8	0.994	20.0	0.526
1.5	0.796	40.0	0.513
3.0	0.659	80.0	0.506
6.0	0.583	∞	0.500

This displacement of the heat flow from the body boundary to its center as the Bi_V criterion increases is illustrated well in table 2 which we composed for the $Es = 4.24$ $(R_1 = 2R_2)$ rectangle according to (25).

Table 2

Point of max. $|\nabla\theta|$ for the $Es = 4.24$ rectangle

$Bi_V = \frac{\alpha}{\lambda}\sqrt{R_1R_2}$	ξ	η	$Bi_V = \frac{\alpha}{\lambda}\sqrt{R_1R_2}$	ξ	η
1.11	1.00	0.726	21.2	0.720	0.323
1.41	0.939	0.650	42.4	0.712	0.309
2.83	0.824	0.493	70.7	0.709	0.303
5.66	0.766	0.399	∞	0.705	0.295
10.6	0.735	0.352			

If $Es = \infty$, i.e., $\frac{R_1}{R_2} = \infty$ and we have, instead of a rectangle, an unlimited strip then the temperature field is one-dimensional and the highest value of the heat flow for all Bi is attained at the $\eta = \pm 1$ boundaries of the strip.

5. According to the meaning of the theorem on the maximum temperature gradient on each curve joining the thermal center of the body to surface points where the temperature equals $\theta_S(\tau)$ exactly or less, a point must exist where the temperature gradient reaches a maximum and this point is certainly located within the body. If we draw thermal current lines from the center of the body to the surface points mentioned then on each of such lines, its maximum temperature gradient will be its maximum heat flow.

The equation of the curve formed from such points of maximum heat flow was derived above [see (19)].

It is understandable that a point from the system (12) lies on the curve (19) corresponding to the Bi value.

The equation of the curve of the maximum $|\nabla\theta|$ for the temperature field of a rectangle in the regular region stage is:

$$2\cot^2\mu_1 \cot^2\mu_2 \tan^2\mu_1\xi \tan^2\mu_2\eta = \cot^4\mu_1 \tan^2\mu_1\xi + \cot^4\mu_2\tan^2\mu_2\eta \tag{31}$$

In the square case, the equation of the curve of heat flow maximums is simplified considerably:

(32) $$2\tan^2\mu\xi \tan^2\mu\eta = \tan^2\mu\xi + \tan^2\mu\eta$$

These curves are shown on figure 1 for the Bi = 1.5, Bi = ∞ and Bi = 3 values. It is easy to note that these curves are homothetic for a square. The linear dimensions of the curves increase as Bi decreases according to the law:

(33) $$\frac{l}{l_\infty} = \frac{\pi}{2\mu}$$

The lines of greatest heat flow for a rectangle with an $\frac{R_1}{R_2} = 2$ ratio of the sides have approximately the same shape (fig. 2), as for a square.

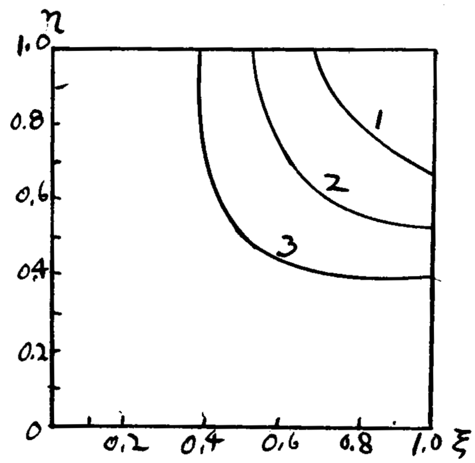

Figure 1
1 - Bi = 1.5 ; 2 - Bi = 3 ; 3 - Bi = ∞

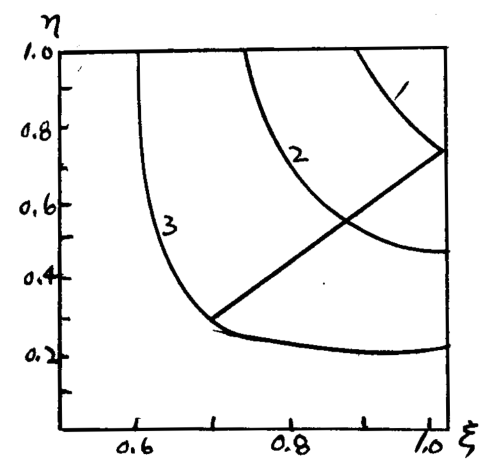

Figure 2
1 - Bi_V = 1.11 ; 2 - Bi_V = 2 ; 3 - Bi_V = ∞

The displacement of the maximum heat flow values to the body center is notable here as the intensity of the heat exchange increases. It is understandable that this approximation is bounded by the $Bi_V = \infty$ line. The curves, located near the rectangle vertex, correspond to the lesser values of Bi_V.

Finally, for Bi_V value determined from (31) when $\xi = \eta = 1$:

(34) $$2 = \cot^2\mu_1 + \cot^2\mu_2$$

the line of maximum heat flow passes only through one point of the rectangle - through its vertex.

For a square, $\mu_1 = \mu_2 = \mu$ and from (34) and (22) there is obtained that for $Bi_V = \frac{\pi}{4}$ that the line of greatest heat flow passes through its vertex.

For a rectangle with the $Es = 4.24$ criterion, the appropriate value is $Bi_V = 0.83$.

January, 1954

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